Twisted cubics on cubic fourfolds and stability conditions

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Setting

A cubic fourfold Y is a smooth cubic hypersurface in $\mathbb{P}^5_{\mathbb{C}}$.
Semiorthogonal decomposition (Kuznetsov):

$$\mathrm{D}^{\mathrm{b}}(Y) = \langle \mathrm{Ku}(Y), \mathcal{O}_Y, \mathcal{O}_Y(1), \mathcal{O}_Y(2) \rangle$$

where $\mathrm{Ku}(Y)$ is the **Kuznetsov component** of Y. $\mathrm{Ku}(Y)$ is a **K3 category**.

Aim and approach

To study the birational geometry of moduli spaces M_d of rational curves of degree d on Y, as moduli spaces of Bridgeland stable objects in $\mathrm{Ku}(Y)$.

Motivations

1) Complete families of polarized hyperkähler varieties.
2) Birational models via wall-crossing.

Key ingredient: in [1], they construct Bridgeland stability conditions on $\mathrm{Ku}(Y)$. We denote such a stability condition by $\bar{\sigma}$.

Degree 1: Lines

The Fano variety $M_1 := F_Y$ of lines in Y is a smooth projective hyperkähler fourfold deformation equivalent to the Hilbert square on a K3 surface (Beauville, Donagi).

Theorem 1

The Fano variety of lines on Y is a moduli space of stable objects in $\mathrm{Ku}(Y)$ with respect to the Bridgeland stability condition $\bar{\sigma}$.

Objects: Consider the ideal sheaf $\mathcal{I}_\ell$ of a line $\ell \subset Y$.

$$\rightsquigarrow F_\ell := \ker(H^0(Y, \mathcal{I}_\ell(1)) \otimes \mathcal{O}_Y \xrightarrow{\mathrm{ev}} \mathcal{I}_\ell) \in \mathrm{Ku}(Y)$$

(Kuznetsov, Markushevich).

Remark 1: Theorem 1 was proved for very general cubic fourfolds in the Appendix of [1] for every $\sigma \in \mathrm{Stab}(\mathrm{Ku}(Y))$. The same argument does not work without the generality assumption. We do the computation with respect to the stability condition $\bar{\sigma}$.
Remark 2: Conic curves in Y? They are residual to lines. $\rightsquigarrow M_2 \to F_Y$ has 3-dimensional fibers.

Degree 3: Twisted cubic curves

Let Y be a cubic fourfold not containing a plane.

$$s : M_3 \to \mathbb{G}(\mathbb{P}^3, \mathbb{P}^5), \quad C \longmapsto \langle C \rangle \cong \mathbb{P}^3$$
$$s^{-1}(\mathbb{P}^3) = \mathrm{Hilb}^{gtc}(S) \longmapsto \mathbb{P}^3$$

where $S = Y \cap \mathbb{P}^3$ is an irreducible reduced cubic surface.
Geometric picture:(Lehn, Lehn, Sorger, van Straten)
1) The morphism above factorizes through a $\mathbb{P}^2$-fibration $M_3 \to M'_Y$, where M'_Y is a smooth and projective variety of dimension eight.
2) The locus of non CM curves in M'_Y is a Cartier divisor D which can be contracted and the resulting variety M_Y is a smooth projective hyperkähler eightfold.

$$\begin{tikzcd}
 & M_3 \arrow[r, "\mathbb{P}^2\text{-fibr.}"] \arrow[d] & M'_Y \arrow[dl] \arrow[d, "\text{contr.}"] & D \cong \mathbb{P}(T_Y) \arrow[l, hook'] \arrow[d] \\
 & \mathbb{G}(\mathbb{P}^3, \mathbb{P}^5) & M_Y & Y \arrow[l, hook']
\end{tikzcd}$$

M_Y is equivalent by deformation to $\mathrm{K3}^{[4]}$ (Addington, Lehn).

Theorem 2

Assume that Y does not contain a plane. Then the LLSvS eightfold M_Y is a moduli space of stable objects in $\mathrm{Ku}(Y)$ with respect to the Bridgeland stability condition $\bar{\sigma}$.

Objects:(Lahoz, Lehn, Macrì, Stellari) Consider the ideal sheaf $\mathcal{I}_{C/S}$ of a twisted cubic curve C in the cubic surface $S \subset Y$.

$$\rightsquigarrow F_C := \ker(H^0(Y, \mathcal{I}_{C/S}(2)) \otimes \mathcal{O}_Y \xrightarrow{\mathrm{ev}} \mathcal{I}_{C/S}(2)).$$

Fact: If C is aCM, then $F_C \in \mathrm{Ku}(Y)$, while in the non CM case $F_C \notin \mathrm{Ku}(Y)$.

$$\rightsquigarrow F'_C := \mathbb{R}_{\mathcal{O}_Y(-1)}(F_C) \in \mathrm{Ku}(Y).$$

Applications

I) Categorical Torelli Theorem

Two cubic fourfolds Y and Y' are isomorphic if and only if there is an equivalence between $\mathrm{Ku}(Y)$ and $\mathrm{Ku}(Y')$ such that the induced isometry between the algebraic Mukai lattices commutes with the degree shift functor.

II) Period point of M_Y

The period point of M_Y is identified with that of the Fano variety of Y.

Derived Torelli Theorem: if Y and Y' are two cubic fourfolds with equivalent Kuznetsov components, then there is a Hodge isometry between their Mukai lattices (Huybrechts).

Question

Two cubic fourfolds with Hodge isometric Mukai lattices have equivalent Kuznetsov components?

The answer is positive for very general cubic fourfolds, for general special cubic fourfolds and for cubic fourfolds having an associated (twisted) K3 surface (Huybrechts, Bayer, Lahoz, Macrì, Nuer, Perry, Stellari).

Idea: $Y \subset M_Y$ and $M_Y \cong \mathfrak{M}_{\sigma'}(v')$ where σ' is a stability conditions on $\mathrm{Ku}(Y')$ $\rightsquigarrow$ $\mathcal{F}$ = restriction of the universal family to $Y \times Y'$.

III) Baby case of Derived Torelli

The following diagram, where i^* is left adjoint of the inclusion $i : \mathrm{Ku}(Y) \hookrightarrow \mathrm{D}^{\mathrm{b}}(Y)$, commutes

$$\begin{tikzcd}
\mathrm{D}^{\mathrm{b}}(Y) \arrow[r, "i^*"] \arrow[d, "\Phi_{\mathcal{F}}"'] & \mathrm{Ku}(Y) \arrow[d, "\mathrm{id}"] \\
\mathrm{D}^{\mathrm{b}}(Y) & \mathrm{Ku}(Y) \arrow[l, "i"]
\end{tikzcd}$$

Idea of proof

I) Recall the construction of $\bar{\sigma}$ in [1].

$$\begin{tikzcd}
 & \tilde{Y} := \mathrm{Bl}_L(Y) \arrow[dl, "\sigma"'] \arrow[dr, "\pi"] & \\
L \subset Y & & \mathbb{P}^3
\end{tikzcd}$$

π is a conic fibration:
$\mathcal{B}_0, \mathcal{B}_1$ = even (resp. odd) part of the sheaf of Clifford algebras associated to π.

$$\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0) = \langle \Psi\sigma^*(\mathrm{Ku}(Y)), \mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3 \rangle.$$

1) For $\alpha > 0$, $\beta \in \mathbb{R}$, consider the weak stability condition $\sigma_{\alpha,\beta} = (\mathrm{Coh}^\beta(\mathbb{P}^3, \mathcal{B}_0), Z_{\alpha,\beta})$ on $\mathrm{D}^{\mathrm{b}}(\mathbb{P}^3, \mathcal{B}_0)$, where:

- $\mathrm{Coh}^\beta(\mathbb{P}^3, \mathcal{B}_0)$ = tilt of $\mathrm{Coh}(\mathbb{P}^3, \mathcal{B}_0)$ with respect to slope β;
- $Z_{\alpha,\beta}(-) = i\mathrm{ch}^\beta_{\mathcal{B}_0,1}(-) + \frac{1}{2}\alpha^2\mathrm{ch}^\beta_{\mathcal{B}_0,0}(-) - \mathrm{ch}^\beta_{\mathcal{B}_0,2}(-)$.

2) Set $\beta = -1$ and tilt again with respect to slope $\mu_{\alpha,-1}$ equal to 0 $\rightsquigarrow \sigma^0_{\alpha,-1}$.
3) The restriction of $\sigma^0_{\alpha,-1}$ to $\mathrm{Ku}(Y)$ defines $\bar{\sigma}$.

II) Given a twisted cubic C, we set $E_C := \Psi\sigma^*(F_C)$, $E'_C := \Psi\sigma^*(F'_C)$.
Proposition 1: E_C is $\sigma_{\alpha,-1}$-stable for $\alpha >> 0$.
Wall-crossing computation: There are three walls.
Proposition 2: At the first wall, E_C remains stable for C aCM, while it becomes unstable for C non CM.
$\rightsquigarrow$ E'_C is stable and it remains stable after crossing the other walls.

Next step (work in progress)

Use the same approach to study twisted quartic curves (or equivalently elliptic quintic curves) on Y.
Expectation: Find the O'Grady's sporadic 10-fold.

References

[1] A. Bayer, M. Lahoz, E. Macrì, P. Stellari, *Stability conditions on Kuznetsov components*, (Appendix joint also with X. Zhao), arXiv:1703.10839.

[2] C. Li, L. Pertusi, X. Zhao, *Twisted cubics on cubic fourfolds and stability conditions*, arXiv:1802.01134.